Análisis monotónico y estabilidad de sistemas de desigualdades de tipo mínimo y de tipo máximo

Monotone analysis and stability of min-type and of max-type inequality systems.

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Joint works with Marco A. López

- Convex Semi-Infinite Programming
- Abstract convexity. Monotonic Analysis.

- Convex Semi-Infinite Programming
  - M.A. López and V.V.de S., "Stability of the feasible set mapping in convex semi-infinite programming", Proceedings of the International Workshop in LSP, Semi-infinite Programming: Recent Advances (Alicante, 1999), 101–120, Nonconvex Optim. Appl., 57, Kluwer Acad. Publ., Dordrecht, 2001. MR1877388 (2003e:90092).

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  - V. Gayá, M. A. López and V.V.de S., "Stability in Convex Semi-Infinite Programming and Rates of Convergence of Optimal Solutions of Discretized Finite Subproblems", Optimization 52 (2003) 6 , 693-713. MR2029012 (2004k:90128).

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  - M.A. López, A.M. Rubinov and V.V. de S. "Stability of the lower level sets of ICAR functions", Numer. Funct. Anal. Optim. 26 (2005), No. 1, 113-127. MR2128747.

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Abstract convex analysis

Abstract convex analysis on a set X deals with some classes L of *elementary* functions defined on X which play the same role that the ordinary linear and affine functions in classical convex analysis. (*Convexity without linearity*)

A function $f : X \to \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$ is called *abstract convex w.r.t. L* (or *L-convex*) if there exists a subset U of L such that

$$f(x) = \sup_{l \in U} l(x)$$

for all $x \in X$.

Min-type functions of the form

$$\langle a, x \rangle := \min_{i \in I} a_i x_i, \qquad x \in \mathbb{R}^n_{++},$$

where $a \in \mathbb{R}^n_{++}$ and $I := \{1, 2, ..., n\}$, are used in monotonic analysis on $\mathbb{R}^n_{++}$ as one of the classes of elementary functions.

ICAR functions

Increasing and convex-along-rays (*ICAR*) functions.

Definition

The function $f : \mathbb{R}^n_{++} \to \mathbb{R} \cup \{+\infty\}$ is an ICAR function on $\mathbb{R}^n_{++}$ if f is increasing on $\mathbb{R}^n_{++}$ (i.e., $x \geq y$ entails $f(x) \geq f(y)$) and the function $f_x :]0, +\infty[\to \mathbb{R} \cup \{+\infty\}$ defined by $f_x(\lambda) := f(\lambda x)$ is convex, for all x.

In *monotonic analysis* these functions play the same relevant role that the usual convex functions in convex analysis.

ICAR functions

If f is a non-constant lower semicontinuous ICAR function (lsc ICAR), then f admits a description as a supremum function

$$f(x) = \sup \left\{ \langle a, x \rangle - b \,\middle|\, \begin{pmatrix} a \\ b \end{pmatrix} \in H \right\}, \tag{1}$$

where $H \subset \mathbb{R}^n_{++} \times \mathbb{R}$ is some non-empty set, and

$$\langle a, x \rangle := \min_{i \in I} a_i x_i.$$

Here a_i is the i-th coordinate of a.

Rubinov, A.M. (2000), *Abstract Convexity and Global Optimization*, Kluwer Academic Publishers, Dordrecht (NL).

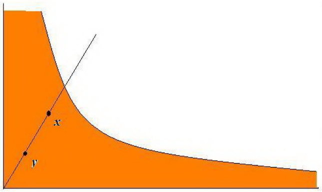
Lower level set

We studied the *stability* of the lower level set

$$F := \{x \in \mathbb{R}^n_{++} \mid f(x) \leq 0\}$$

of a finite-valued ICAR function f.

F is a closed (relatively to $\mathbb{R}^n_{++}$) *normal set*: F is such that $x \geq y$ and $x \in F$ entail $y \in F$.



Parameter space

Stability concerns the *quality of the representation* of the set F by means of the function f.

The *parameter space* is the family $\mathcal{F}$ of all the finite-valued ICAR functions on $\mathbb{R}^n_{++}$ embedded, in principle, with the topology induced by the pointwise convergence of functions.

The main object of our approach is the *solution set mapping* $\mathcal{S} : \mathcal{F} \rightrightarrows \mathbb{R}^n_{++}$ defined by

$$\mathcal{S}(f) := \{x \in \mathbb{R}^n_{++} \mid f(x) \leq 0\} \equiv F.$$

$\mathcal{F}_c = \operatorname{dom} \mathcal{S}$ ($f \in \mathcal{F}_c$ if and only if $\mathcal{S}(f) \neq \emptyset$).

Properties of an ICAR function.

- An ICAR function on $\mathbb{R}^n_{++}$, f, is continuous on $\operatorname{int}\operatorname{dom} f$.

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- If f is a lsc ICAR function on $\mathbb{R}^n_{++}$, $x \in \mathbb{R}^n_{++}$, and $f_x(\lambda) := f(\lambda x)$, then
  - f_x is an increasing lsc convex function.
  - f_x is either strictly increasing in its effective domain, or there exists a point λ_x such that $f_x(\lambda) = \min\{f(z), z \in \mathbb{R}^n_{++}\}$ for every $\lambda \in]0, \lambda_x]$ and f_x is strictly increasing in $(\operatorname{dom} f_x) \cap]\lambda_x, +\infty[$.

Properties of an ICAR function.

The following result is crucial in deriving stability results for the solution set mapping $\mathcal{S}$.

Theorem

Let f be an ICAR function on $\mathbb{R}^n_{++}$. Then, the function $x \mapsto f^{'}(x, x)$ is upper semicontinuous (usc, in short) at any point of $\operatorname{int}\operatorname{dom} f$.

Properties of an ICAR function.

Lemma

Let f be an ICAR function on $\mathbb{R}^n_{++}$. Let $\Omega \subset \operatorname{int}\operatorname{dom} f$ be a set such that there exists a number $a > 0$ with the property $a\mathbf{1} \leq x$ for each $x \in \Omega$. Assume that $C = \sup_{y\in\Omega} f'(y, y) < +\infty$. Then

$$|f(x) - f(y)| \leq \frac{C}{a} \|x - y\|,$$

for all $x, y \in \Omega$.

Theorem

Let f be an ICAR function on $\mathbb{R}^n_{++}$. Then f is locally Lipschitz on $\operatorname{int}\operatorname{dom} f$.

Convergence of ICAR functions

Theorem

Let $\{f_k\}$ *be a sequence of finite-valued ICAR functions on* $\mathbb{R}^n_{++}$ *such that* $f_k(x) \to f(x)$ *for each* $x \in \mathbb{R}^n_{++}$*, where* f *is a finite-valued ICAR function on* $\mathbb{R}^n_{++}$*. Let* Ω *be a compact set in* $\mathbb{R}^n_{++}$*. Then there exists a constant* C *such that*

$$\sup_{x\in\Omega} f'(x,x) \le C$$

and

$$\sup_{x\in\Omega} f'_k(x,x) \le C,$$

for each $k = 1, 2, \dots$.

Convergence of ICAR functions

Pointwise convergence and uniform covergence on compact sets are equivalent for ICAR functions.

Theorem

Let $\{f_k\}$ be a sequence of finite-valued ICAR functions on $\mathbb{R}^n_{++}$ such that $f_k(x) \to f(x)$ for each $x \in \mathbb{R}^n_{++}$, where f is a finite-valued ICAR function on $\mathbb{R}^n_{++}$. Let Ω be a compact set in $\mathbb{R}^n_{++}$. Then $f_k \to f$ uniformly on Ω.

A distance in the parameter space

We start from a sequence of compact sets in $\mathbb{R}^n_{++}$, $\{K_p\}$, such that $K_p \subset \operatorname{int} K_{p+1}$ and $\mathbb{R}^n_{++} = \bigcup_{p=1}^{\infty} K_p$.
(For instance, $K_p := \{x \in \mathbb{R}^n_{++} \mid \frac{1}{k}\mathbf{1} \leq x \leq k\mathbf{1}\}$, $p = 1, 2, ...$)

Then, given two functions in $\mathcal{F}$, f and g, let δ_p be the (pseudo-) metric

$$\delta_p(f, g) := \max_{x \in K_p} \{|f(x) - g(x)|\}, \tag{2}$$

for $p = 1, 2, ...$

We define the distance δ as

$$\delta(f, g) := \sum_{p=1}^{\infty} \frac{1}{2^p} \frac{\delta_p(f, g)}{1 + \delta_p(f, g)}. \tag{3}$$

Stability results

Definition

$f \in \mathcal{F}_c$ is *stable with respect to the consistency* if $f \in \operatorname{int} \mathcal{F}_c$.

Generically, in optimization, the stability of the feasible set map addresses to the existence of *strict feasible solutions*.

Theorem

$f \in \operatorname{int} \mathcal{F}_c$ if and only if a point $x^0 \in \mathbb{R}^n_{++}$ exists such that $f(x^0) < 0$.

Other stability criteria

- $\mathcal{S}$ is *lower semicontinuous in the Berge sense* (*B-lsc*) at $f \in \mathcal{F}_c$ if, for each open set $W \subset \mathbb{R}^n_{++}$ such that $\mathcal{S}(f) \cap W \neq \emptyset$, there exists an open set $U \subset \mathcal{F}$ containing f, such that $\mathcal{S}(f_1) \cap W \neq \emptyset$ for every $f_1 \in U$.

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- $\mathcal{S}$ is *closed* at $f \in \mathcal{F}_c$ if for all sequences $\{f_k\} \subset \mathcal{F}_c$ and $\{x^k\} \subset \mathbb{R}^n_{++}$ satisfying $\lim_k f_k = f$, $\lim_k x^k = \widehat{x} \in \mathbb{R}^n_{++}$ and $x^k \in \mathcal{S}(f_k)$, one has $\widehat{x} \in \mathcal{S}(f)$.

Bouligand continuity

Relatively to $\mathcal{S}$ we can consider the

$$\textit{inner limit} \; \liminf_{\widehat{f} \to f} \mathcal{S}\left(\widehat{f}\right)$$

which is the set of points that are limit points of all possible sequences $\{x^r\}$, $x^r \in \mathcal{S}(f_r)$, for all $\{f_r\}$, $f_r \to f$.

The

$$\textit{outer limit} \limsup_{\widehat{f} \to f} \mathcal{S}\left(\widehat{f}\right)$$

consists of all possible cluster points of such sequences.

When $\mathcal{S}(f) = \limsup_{\widehat{f} \to f} \mathcal{S}\left(\widehat{f}\right)$ it is said that $\mathcal{S}$ is *outer semicontinuous (osc)* at f and, similarly, $\mathcal{S}$ is *inner semicontinuous (isc)* at f if $\mathcal{S}(f) = \liminf_{\widehat{f} \to f} \mathcal{S}\left(\widehat{f}\right)$.

$\mathcal{S}$ is *continuous in the Bouligand sense* at $f \in \mathcal{F}_c$ if $\mathcal{S}$ is simultaneously osc and isc at f.

Bouligand continuity

Proposition

The solution set mapping $\mathcal{S}$ is a closed mapping at every $f \in \mathcal{F}_c$.

Proposition

Let $f \in \mathcal{F}_c$. Then $f \in \operatorname{int} \mathcal{F}_c$ if and only if the feasible set mapping $\mathcal{S}$ is B-lsc at f, if and only if $\mathcal{S}$ is continuous in the Bouligand sense at f.

Upper semicontinuity

The upper semicontinuity in the sense of Berge is a very difficult property to be held.

Actually, the following example shows that the boundedness of F is even not sufficient for the upper semicontinuity of $\mathcal{S}$ at $f \in \mathcal{F}_c$.

Example

Consider the finite-valued ICAR functions on $\mathbb{R}^2_{++}$:

$$f_k(x) := \left(x_1 - \frac{1}{k}\right)_+ (x_2 - 1)_+ + (x_1 - 1)_+ \left(x_2 - \frac{1}{k}\right)_+, \ k = 1, 2, \ldots$$

Obviously

$$f_k(x) \to f(x) := x_1 (x_2 - 1)_+ + (x_1 - 1)_+ x_2.$$

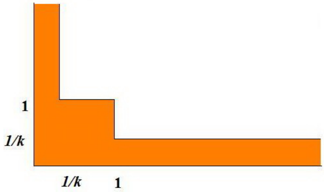
Moreover

$$\mathcal{S}(f) = \{x \in \mathbb{R}^2_{++} \mid x \leq \mathbf{1}\},$$

which is bounded, whereas

$$f_k(1/k, k) = 0, \ k = 1, 2, \ldots;$$

i.e., $\mathcal{S}(f_k)$ is unbounded, precluding the upper semicontinuity of $\mathcal{S}$ at $f \in \mathcal{F}_c$.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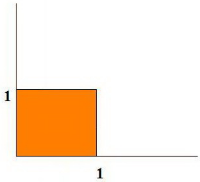
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Metric regularity

Associated with the function $f \in \mathcal{F}$ we introduce the set valued function $\mathcal{M} : \mathbb{R}^n_{++} \rightrightarrows \mathbb{R}$ defined by

$$\mathcal{M}(x) := f(x) + \mathbb{R}_+.$$

Given the scalar β we have $\beta \in \mathcal{M}(z)$ if and only if z satisfies the inequality $f(x) \leq \beta$. Equivalently,

$$\mathcal{M}^{-1}(\beta) = \{z \in \mathbb{R}^n_{++} \mid f(z) \leq \beta\}.$$

In other words, $\mathcal{M}^{-1}$ coincides with $\mathcal{S}$ but restricted to consider functions that differ from f only by an additive constant.

Metric regularity

Definition

Given $f \in \mathcal{F}_c$ and $z \in \mathcal{S}(f)$, we say that $\mathcal{M}$ is *metrically regular at* z *for* 0 if there exist three positive scalars k, ε and ρ such that

$$d(y, \mathcal{M}^{-1}(\beta)) \leq kd(\beta, \mathcal{M}(y)), \tag{4}$$

provided that

$$\|y - z\| < \varepsilon \text{ and } |\beta| < \rho. \tag{5}$$

It can easily be checked that

$$d(\beta, \mathcal{M}(y)) = (f(y) - \beta)^+,$$

with $a^+ := \max\{a, 0\}$.

Metric regularity

Proposition

Let $f \in \mathcal{F}_c$. *Then* $f \in \operatorname{int} \mathcal{F}_c$ *if and only if* $\mathcal{M}$ *is metrically regular at* z *for* 0, *for every* $z \in F$.

Min-type functions

$$\langle a, x\rangle := \min_{i\in I} a_i x_i, \qquad x \in \mathbb{R}^n_{++},$$

where $a \in \mathbb{R}^n_{++}$ and $I := \{1, 2, ..., n\}$.

Semi-infinite systems

$$\sigma := \{\langle a_t, x\rangle \leq b_t,\ t \in T\}$$

of *min-type inequalities* arise in monotonic analysis.

They describe the *normal* subsets of $\mathbb{R}^n_{++}$ that are abstract convex w.r.t. to the class of min-type functions.

Different elementary functions

Min-type functions $l_a(x) := \langle a, x \rangle$ can be considered as *abstract linear functions* in the model under consideration. Denote by L the set of all such functions.

Functions of the form $h_{a,b}(x) := l_a(x) - b$, where $a \in \mathbb{R}^n_{++}$ and $b \in \mathbb{R}$, play the role of *abstract affine functions.* Let
$H = \{h_{a,b} : a \in \mathbb{R}^n_{++}, b \in \mathbb{R}\}$.

A function $f : \mathbb{R}^n_{++} \to \mathbb{R}_{+\infty}$ is

- L-convex if and only if f is IPH (increasing and positively homogeneous of degree one).
- H-convex if and only if f is a lower semicontinuous ICAR (increasing and convex-along-rays) function.

IPH (lsc ICAR) functions in this model are similar to lsc sublinear functions (lsc convex functions) in classical convex analysis.

Different elementary functions

Other different model uses as elementary functions those having the form

$$l(x) := \min(\langle a, x\rangle, c), \qquad x \in \mathbb{R}^n_{++},$$

where $a \in \mathbb{R}^n_{++}$ and c is a positive number.

A function f is abstract convex w.r.t. this new class of elementary functions if and only if f is *ICR* (increasing and co-radiant).

A function f is called *co-radiant* if $f(\alpha x) \geq \alpha f(x)$ for $\alpha \in (0, 1]$ and $x \in \mathbb{R}^n_{++}$.

The main objective of this paper is then to study the *stability* of the set

$$F = \{x \in \mathbb{R}^n_{++} \mid f(x) \leq 0\},$$

zero-lower level set of an abstract convex function f depending on its representation as a *solution set* of some associated *abstract linear semi-infinite system*.

We considered two models of monotonic analysis on $\mathbb{R}^n_{++}$: systems of the form

$$\sigma := \{\langle a_t, x\rangle \leq b_t,\ t \in T\},$$

called *min-type systems of the first type*, and systems

$$\rho := \{\min(\langle a_t, x\rangle, c_t) \leq b_t,\ t \in T\},$$

called *min-type systems of the second type*.

Perturbations

The stability of the solution set of the system $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in T\}$ is studied under small perturbations of *all* the coefficients involved in the system.

Perturbations of the coefficients in the *nominal system* σ yield a new *perturbed system* $\sigma_1 := \{\langle a_t^1, x\rangle \leq b_t^1,\ t \in T\}$, and the associated *perturbed solution set* F_1.

The perturbations should be sufficiently small to guarantee $\{a_t^1, t \in T\} \subset \mathbb{R}^n_{++}$.

Associated with σ_1 we have a *perturbed lsc ICAR function*

$$f_1(x) := \sup\{\langle a_t^1, x\rangle - b_t^1,\ t \in T\}$$

and

$$F_1 = \{x \in \mathbb{R}^n_{++} \mid f_1(x) \leq 0\}.$$

Measure of the perturbations

Parameter space: the set Θ of all the min-type inequality systems of the first type on $\mathbb{R}^n_{++}$, with a fixed index set T,

$$\Theta = \left(\mathbb{R}^n_{++} \times \mathbb{R}\right)^T$$

We define the *extended distance* $d : \Theta \times \Theta \to [0, +\infty]$

$$d\left(\sigma_1, \sigma\right) := \sup_{t \in T} \left\| \begin{pmatrix} a_t^1 \\ b_t^1 \end{pmatrix} - \begin{pmatrix} a_t \\ b_t \end{pmatrix} \right\|. \tag{6}$$

Fenchel-Moreau conjugate function

The *Fenchel-Moreau conjugate* of f is given by

$$f^{*}(a) := \sup\left\{\langle a, x\rangle - f(x) \mid x \in \mathbb{R}^{n}_{++}\right\},$$

where $a \in \mathbb{R}^{n}_{++}$. The conjugate f^{*} is a lsc ICAR function.

$$F = \{x \in \mathbb{R}^{n}_{++} \mid f(x) \leq 0\} = \{x \in \mathbb{R}^{n}_{++} \mid \langle a, x\rangle \leq f^{*}(a),\ a \in \operatorname{dom} f^{*}\}.$$

If $\binom{a}{b} \in \mathbb{R}^n_{++} \times \mathbb{R}$, then $\frac{b}{a}$ denotes the vector in $\mathbb{R}^n$ whose $i-$th coordinate is $\frac{b}{a_i}$.

Gale-type theorem:

Theorem

The following three assertions are equivalent:
(i) The system $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in T\}$ *is inconsistent* $(i.e., \sigma \notin \Theta_c)$;
(ii) $\left\{\frac{b_t}{a_t}, t \in T\right\} + \mathbb{R}^n_{++} = \mathbb{R}^n_{++}$;
(iii) $\mathbf{0} \in \mathrm{cl}\left\{\frac{b_t}{a_t}, t \in T\right\}$.

Farkas-type lemma:

Theorem

The inequality $\langle a, x\rangle \leq b$ *(with* $a \in \mathbb{R}^n_{++}$ *and* $b > 0$*) is a consequence of the system* $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in T\}$ *if and only if*

$$\frac{b}{a} \in \operatorname{cl}\left(\left\{\frac{b_t}{a_t}, t \in T\right\} + \mathbb{R}^n_{++}\right). \tag{3.1}$$

Corollaries

- $\langle a, x\rangle \le b$ is a consequence of the system σ if and only if $\frac{b}{a} \in \mathrm{cl}(\mathbb{R}^n_{++} \setminus F)$.

- $\langle a, x\rangle \le b$ is a consequence of the system $\sigma = \{\langle a_t, x\rangle \le b_t,\ t \in T\}$ if and only if

$$b \ge \inf_{t\in T} \left(\max_{i\in I} \frac{a_i}{a_{ti}} \right) b_t .$$

Stability with respect to the consistency

Proposition

Let $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in T\} \in \Theta_c$. *Then* $\sigma \in \operatorname{int}\Theta_c$ *if and only if* $\inf\{b_t, t \in T\} > 0$.

Stability with respect to the consistency

Definition

The system $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in T\}$ is said to satisfy the *strong Slater condition (SS condition)* if there exist x^0 and $\eta > 0$ such that $\left\langle a_t, x^0 \right\rangle \leq b_t - \eta$, for all $t \in T$. In this case x^0 is called an *SS point.*

Proposition

$\sigma \in \operatorname{int} \Theta_c$ *if and only if* σ *satisfies the strong Slater condition.*

Stability with respect to the consistency

The *feasible set mapping* $\mathcal{F} : \Theta \rightrightarrows \mathbb{R}^n_{++}$, assigns to each $\sigma \in \Theta$ its solution set F; $\mathcal{F}(\sigma) = F$.

Proposition

Let $\sigma \in \Theta_c$. Then $\sigma \in \operatorname{int} \Theta_c$ if and only if the feasible set mapping $\mathcal{F}$ is B-lsc at σ, if and only if $\mathcal{F}$ is continuous in the Bouligand sense at σ.

Metric regularity

Associated with the *fixed* function $a : T \longrightarrow \mathbb{R}^n_{++}$, we have the set valued function $\mathcal{M} : \mathbb{R}^n_{++} \rightrightarrows \mathbb{R}^T_{++}$ defined by

$$\mathcal{M}(x) := \langle a(.), x \rangle + \mathbb{R}^T_{+}.$$

Let $b : T \longrightarrow \mathbb{R}_{++}$. Then $b \in \mathcal{M}(z)$ if and only if z is a solution of the system $\{\langle a_t, x \rangle \leq b_t,\ t \in T\}$.
$\mathcal{M}^{-1}$ is the feasible set mapping restricted to systems with fixed $a_t,\ t \in T$.

Definition

The system $\sigma = \{\langle a_t, x \rangle \leq b_t,\ t \in T\}$ satisfies the *Robinson-Ursescu condition* if $b \in \operatorname{int} \mathcal{M}(\mathbb{R}^n_{++})$.

Metric regularity

Proposition

Given $\sigma \in \Theta_c$, $\sigma \in \operatorname{int}\Theta_c$ *if and only if* σ *satisfies the Robinson-Ursescu condition.*

Proposition

Let $\sigma = \{\langle a_t, x\rangle \leq b_t,\ t \in T\} \in \Theta_c$. *Then* $\sigma \in \operatorname{int}\Theta_c$ *if and only if* $\mathcal{M}$ *is metrically regular at* z *for* b*, for every* $z \in F$.

Stability of a finite ICAR function

Proposition

Let us consider a non-constant finite ICAR function f on $\mathbb{R}^n_{++}$ such that $\inf \{f(x), x \in \mathbb{R}^n_{++}\} < 0$.
Then, there exists a representation of its lower level set $F = \{x \in \mathbb{R}^n_{++} : f(x) \leq 0\}$ *as a solution set of a semi-infinite min-type system of the first type in $\mathbb{R}^n_{++}$ which is stable in a certain parameter space.*

Max-type functions

max-type functions:

$$\langle a, x \rangle := \max_{i \in I} a_i x_i, \qquad x \in \mathbb{R}^n_+,$$

where $a \in \mathbb{R}^n_+$ and $I := \{1, 2, ..., n\}$.

Semi-infinite systems $\sigma := \{\langle a_t, x \rangle \geq b_t,\ t \in T\}$ of *max-type inequalities* arise in monotonic analysis describing the so-called *co-normal* subsets of $\mathbb{R}^n_+$.

If f is any increasing function defined on $\mathbb{R}^n_+$, then the upper level sets $\{x \in \mathbb{R}^n_+ : f(x) \geq c\}$ are co-normal. We studied the stability of the upper level set, $F = \{x \in \mathbb{R}^n_+ : f(x) \geq 0\}$,

Example

For instance, a continuous increasing positively homogeneous function f can be represented as the pointwise infimum of a subset of functions of the form

$$\langle a,x \rangle - b,$$

for some $a \in \mathbb{R}^n_+$, $b \in \mathbb{R}$, so it is an abstract concave function, e.g. $f(x) = \inf \{ \langle a_t,x \rangle - b_t,\ t \in T \}$, for some (possibly infinite) index set T.

The upper level set F of the function f is the solution set of the max-type system

$$\sigma := \{ \langle a_t,x \rangle \geq b_t,\ t \in T \}.$$

Co-normal sets

A set $U \subset \mathbb{R}^n_+$ is *co-normal* if

$$x \in U, \; y \in \mathbb{R}^n_+ \text{ and } y \geq x \;\Rightarrow\; y \in U.$$

- The empty set is co-normal by definition.

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- A nonempty co-normal set U is *co-radiant*; i.e., if $x \in U$ and $\lambda \geq 1$, then $\lambda x \in U$.

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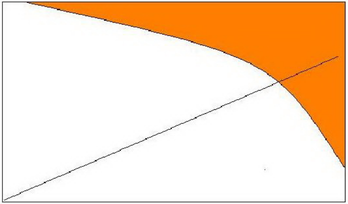
- The intersection and the union of a family of co-normal sets are co-normal. The solution set of $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\}$ is closed and co-normal.
- A nonempty co-normal set U is *co-radiant*; i.e., if $x \in U$ and $\lambda \geq 1$, then $\lambda x \in U$.
- For each $y > \mathbf{0}$ a positive scalar λ exists such that $\lambda y \in U$.



Parameter space

We consider as a *parameter space* the set Θ of all the max-type inequality systems on $\mathbb{R}^n_+$, with a fixed index set T.

Θ can be identified with $\left(\mathbb{R}^{n+1}_+\right)^T$, since each system σ can be represented by $\left\{\binom{a_t}{b_t}\right\}_{t\in T}$

Notation: If $a \in \mathbb{R}^n_+$ and $b \in \mathbb{R}_+$ then $\frac{b}{a}$ is the vector in $\overline{\mathbb{R}}^n_+$ whose i-th coordinate is $\frac{b}{a_i}$, where we adopt the convention

$$\frac{b}{0} = \begin{cases} 0, & \text{if } b = 0, \\ +\infty, & \text{if } b > 0; \end{cases} \tag{7}$$

Gale-type theorem:

Proposition

The system $\sigma := \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$ *if and only if* $\sup\left\{\frac{b_t}{\|a_t\|}, t \in T\right\} < \infty.$

(Here the supremum is taken in $\overline{\mathbb{R}}_+$*.)*

Farkas-type theorem

Farkas' lemma.

Proposition

The inequality $\langle a, x\rangle \geq b$ *is a consequence of the system* $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\}$ *if and only if*

$$\frac{b}{a} \in \operatorname{cl} \bigcup_{t\in T} \left\{ u \,\middle|\, \mathbf{0} \leq u \leq \frac{b_t}{a_t} \right\}, \tag{8}$$

where the closure is taken in $\overline{\mathbb{R}}^n_+$.

Proposition

Let $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$. *Then* $\sigma \in \operatorname{int}\Theta_c$ *if and only if*

$$\mathbf{0} \notin \operatorname{cl}\{a_t,\ t \in T\}.$$

Conditions for stability

Proposition

$\sigma \in \operatorname{int}\Theta_c$ if and only if σ satisfies the strong Slater condition.

Proposition

If $\sigma \in \operatorname{int}\Theta_c$ and $z > \mathbf{0}$ is feasible for σ, then λz is an SS point for all $\lambda > 1$.

If $\sigma \in \operatorname{int}\Theta_c$ *and* $z \in \operatorname{int} F$*, then* z *is an SS point for* σ*.*

Remark

If $x^0 > \mathbf{0}$ *is SS point for* σ*, then* x^0 *does not need to belong to* $\operatorname{int} F$*.*

Conditions for stability

Proposition

Let $\sigma \in \Theta_c$. Then $\sigma \in \operatorname{int} \Theta_c$ if and only if the feasible set mapping $\mathcal{F}$ is B-lsc at σ, if and only if $\mathcal{F}$ is continuous in the Bouligand sense at σ.

Conditions for stability

Definition

The system $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\}$ satisfies the *Robinson-Ursescu condition* if $b \in \operatorname{int} \mathcal{M}(\mathbb{R}^n_+)$.

Proposition

Given $\sigma \in \Theta_c$, $\sigma \in \operatorname{int} \Theta_c$ if and only if σ satisfies the Robinson-Ursescu condition.

Proposition

Let $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$. Then $\sigma \in \operatorname{int} \Theta_c$ if $\mathcal{M}$ is metrically regular at z for b, for every $z \in F$.

In our context, it can be easily checked that

$$d(b^1, \mathcal{M}(y)) = \sup_{t \in T} (b_t^1 - \langle a_t, y \rangle)^+,$$

with $a^+ := \max\{a, 0\}$.

Theorem

Let δ *and* γ *be any pair of positive real numbers. If* $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$ *is such that* $\|a_t\| \geq \delta$ *for all* $t \in T$, *then*

$$d\left(y, \mathcal{M}^{-1}\left(b^1\right)\right) \leq \frac{1}{\delta\gamma} d\left(b^1, \mathcal{M}\left(y\right)\right), \tag{9}$$

for any $b^1 \in \mathbb{R}_+^T$ *and for all* $y \in \mathbb{R}_+^n$ *with* $\frac{\min_i y_i}{\max_i y_i} > \gamma$. *Moreover, for* $y = \mathbf{0}$ *it holds:*

$$d\left(\mathbf{0}, \mathcal{M}^{-1}\left(b^1\right)\right) \leq \frac{1}{\delta} d\left(b^1, \mathcal{M}\left(\mathbf{0}\right)\right).$$

Corollary

Let $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \operatorname{int}\Theta_c$, *then* $\mathcal{M}$ *is metrically regular at* z *for* b, *for every* $z \in F$ *such that* $z > \mathbf{0}$.

Global error bound

Proposition

If $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \Theta_c$ *and*

$$\inf\{a_{t_i} : i \in I_+(a_t), t \in T\} = \gamma > 0,$$

then the solution set F *has a global error bound on* $\mathbb{R}^n_+$ *with bound* $k \leq \gamma^{-1}$.

Metric regularity

Theorem

If $\sigma = \{\langle a_t, x\rangle \geq b_t,\ t \in T\} \in \operatorname{int}\Theta_c$ *and*

$$\inf\left\{a_{t_i} : i \in I_+\left(a_t\right), t \in T\right\} = \gamma > 0,$$

then

$$d\left(y, \mathcal{M}^{-1}\left(b^1\right)\right) \leq \frac{1}{\gamma} d\left(b^1, \mathcal{M}\left(y\right)\right),$$

for any $b^1 \in \mathbb{R}_+^T$ *and all,* $y \in \mathbb{R}_+^n$*. In particular,* $\mathcal{M}$ *is metrically regular at any* $y \in F$ *for* b*.*

Corollary

If T *is finite and* σ *is consistent, then* $\sigma \in \operatorname{int}\Theta_c$ *if and only if* $\mathcal{M}$ *is metrically regular at any* $z \in F$ *for* b*.*